Name ____ Test 3, Spring 2021

1) Find all the eigenspaces of the matrix below (15 points)

$$\begin{bmatrix} -1 & -3 & 3 \\ 0 & -1 & 0 \\ 0 & -3 & 2 \end{bmatrix}$$

$$|xI - A| = \begin{vmatrix} x+1 & 3 & -3 \\ 0 & x+1 & 0 \\ 0 & 3 & x-2 \end{vmatrix} = (x+1)\begin{vmatrix} x+1 & 0 \\ 3 & x-2 \end{vmatrix} = (x+1)(x+1)(x-2)$$

For $\lambda = -1$ we get:

$$\begin{bmatrix} -1+1 & 3 & -3 \\ 0 & -1+1 & 0 \\ 0 & 3 & -1-2 \end{bmatrix} = \begin{bmatrix} 0 & 3 & -3 \\ 0 & 0 & 0 \\ 0 & 3 & -3 \end{bmatrix} \sim_R \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has eigenspace $span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right\}$.

For $\lambda = 2$ we get:

$$\begin{bmatrix} 2+1 & 3 & -3 \\ 0 & 2+1 & 0 \\ 0 & 3 & 2-2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & -3 \\ 0 & 3 & 0 \\ 0 & 3 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has eigenspace $span\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

2) Find the diagonalizaion of the matrix from the previous problem. (5 points)
(If you couldn't solve the previous problem, make up an answer to answer this problem)

$$\begin{bmatrix} -1 & -3 & 3 \\ 0 & -1 & 0 \\ 0 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1}$$

3) Given the basis below, find an orthogonal basis for the same vector space. (10 points)

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}2\\3\\4\end{bmatrix}\right\}$$

The first vector is $\begin{bmatrix}1\\1\\0\end{bmatrix}$

The second vector is $\begin{bmatrix}2\\3\\4\end{bmatrix} - proj_{\begin{bmatrix}1\\1\\0\end{bmatrix}}\left(\begin{bmatrix}2\\3\\4\end{bmatrix}\right)$

$$proj_{\begin{bmatrix}1\\1\\0\end{bmatrix}}\left(\begin{bmatrix}2\\3\\4\end{bmatrix}\right) = \frac{2+3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2.5\\2.5\\0\end{bmatrix}$$

$$\begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}2.5\\2.5\\0\end{bmatrix} = \begin{bmatrix}-0.5\\0.5\\4\end{bmatrix}$$

The orthogonal basis is:

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}-0.5\\0.5\\4\end{bmatrix}\right\}$$

4) Answer the following questions. (3 points each)

A) Let A be a 5×5 with eigenvalues $0, 0, 1, 2, 3$. What is the maximum rank of A?

4

B) Let A be a 3×5 matrix whose nullity is 4. When row reduced, how many rows of zeroes are there?

2

C) Consider a system of 4 equations and 4 variables that has a unique solution. When row reduced, how many pivots does the corresponding matrix have?

4

D) Let A be a 6×6 matrix whose corresponding linear transformation T is onto. Is T one-to-one?

Yes

A) Let A be a 3×3 matrix whose corresponding linear transformation T is not one-to-one. What is the determinant of A?

0

5) Given the basis and vector below, find a formula for the vector $\vec{x}$ in standard coordinates. (10 points)

$$B = \left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}4\\2\end{bmatrix}\right\}; [\vec{x}]_B = \begin{bmatrix}5\\3\end{bmatrix}_B$$

$$[\vec{x}]_S = 5\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}4\\2\end{bmatrix}$$

6) Find a formula for for $[T]$ given the facts below. (5 points)

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}5\\6\end{bmatrix}; T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}7\\8\end{bmatrix}$$

$$[T] = \begin{bmatrix}5 & 7\\6 & 8\end{bmatrix}\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}^{-1}$$

(Set up the diagram to make this easy, don't try to memorize a formula)

7) Find the null space of the matrix below. (10 points)

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$span\left\{\begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

8) Reduce the matrix below to reduced echelon form. (10 points)

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 3 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 3 & 4 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 6 \\ 0 & 1 & 3 & 4 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & -17 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}$$

9) Let A be a 5×5 matrix with eigenvalues $0, 1, 2, 3, 4$. Is A is diagonalizable? (5 points)

Yes

10) Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ and $f(x) = x^2 + 5$. Find $f(A)$. (5 points)

$$f(A) = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}^2 + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 0 & 14 \end{bmatrix}$$

11) Find the product below. (5 points)

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 3 & 1 & 3 & 3 & 3 \\ 4 & 4 & 1 & 4 & 4 \\ 5 & 5 & 5 & 1 & 5 \\ 6 & 6 & 6 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -2 & -2 & -2 & -2 \\ 3 & 1 & 3 & 3 & 3 \\ 14 & 14 & 11 & 6 & 14 \\ 5 & 5 & 5 & 1 & 5 \\ 6 & 6 & 6 & 6 & 1 \end{bmatrix}$$

12) Find the inverse of the matrix below. (5 points)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \end{bmatrix} \sim_R \begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}$$